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Analyticity and Mixing Properties for Random Cluster Model with $q > 0$ on $\mathbb{Z}^d$

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We study the Random Cluster Model on $\mathbb{Z}^d$ for p near either 0 or 1 and for all $q > 0$ and we prove by mean of cluster expansion methods the analyticity of the pressure and finite connectivities in both regimes. These results are valid also in the regime $q < 1$ and they imply that percolation probability is strictly less than 1.

KEY WORDS: Random cluster model, cluster expansion

1. INTRODUCTION

The Random Cluster Model (RCM) is a stochastic process introduced in the early '70 by Fortuin and Kastelyn[(4)] which had a very relevant impact in probability and statistical mechanics. The process is initially defined on a finite graph. Each edge of the graph can be open or closed, and the process depends on two parameters, namely p and q, representing respectively the weight of each open edge of the graph and the weight of each connected component of open edges of the graph. The process is then defined on a countably infinite graph, by studying the limit of suitably chosen sequences of finite sub-graphs with suitably chosen boundary conditions. Varying the parameters p and q some of the most popular systems in statistical mechanics (e.g. Ising and Potts model) and probability (e.g. Bernoulli percolation) may be recovered.

The RCM has been mainly investigated when the underlying graph is the regular cubic lattice $\mathbb{Z}^d$, but during the last decade a growing interest about RCM and related statistical mechanics systems on general graphs has emerged.

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Few results on RCM can be proved for all the values of the parameters q and p. In particular, the existence of the pressure, its independency on boundary conditions and its differentiability have been proved in ref. 6 for $\mathbb{Z}^d$, and for a certain class of general graphs in ref. 10. This shows that the whole machinery of the statistical mechanics, and its probabilistic counterpart, can be used for all the values of the parameters of the RCM. However the study of the statistical mechanics properties of RCM has been developed so far only in the region $q \geq 1$ where the powerful tool given by the so-called FKG inequalities is available. In particular, by comparison inequalities (see refs. 1, 4 and 5), is possible to prove that, for $q \geq 1$, it exists a critical value $p_c(q) < 1$ such that for $p < p_c(q)$ the probability to have a infinite open cluster is zero, while for $p > p_c(q)$ is one (ref. 1, Theorem 4.2). Many other important results can be collected for the RCM on $\mathbb{Z}^d$ in the regime $q \geq 1$. We refer the reader to the monograph[5] for a detailed description of these results and references. We list here only the results about the exponential decay of connectivity because they are directly related with our results. In the supercritical phase (up to the slab percolation threshold in $d \geq 3$), the exponential decay of finite connectivities follows from the renormalization group techniques developed in ref. 2. Concerning the p small regime, the exponential decay of connectivities can be obtained by comparison inequalities (see e.g. Theorem 3.2 in refs. 5) and using the known results on Bernoulli bond percolation and/or Potts model.

None of the latter results has been proved when $q < 1$, since the FKG inequalities are not valid anymore in this regime. It is possible to use the comparison inequalities also in the region $q < 1$ to obtain exponential bounds on the decay of the connectivity functions for p sufficiently small (see ref. 1, 5 and 7). Note however that the same device wouldn't work to derive the exponential decay in the $q < 1$ and p large regime for finite connectivities, since these are neither increasing nor decreasing functions (in the FKG sense).

In this paper we study the statistical mechanics behavior of the RCM $\mathbb{Z}^d$ ($d \geq 2$) for p near either 0 or 1 and for all $q > 0$. We prove by mean of cluster expansion methods the analyticity of the pressure and finite connectivities in both regimes. Namely, for the subcritical regime we obtain that, for any fixed value of $q > 0$ there is $\varepsilon_q > 0$ such that, for p in the disk $|p| < \varepsilon_q$, the pressure and n-point connectivity functions of the RCM on $\mathbb{Z}^d$ exist and can be written explicitly as analytic functions of p. For the supercritical regime we obtain that, for any fixed value of $q > 0$ there is $\delta_q > 0$ such that for any p in the disk $|1 - p| < \delta_q$, the pressure and the n-point finite connectivity functions of the RCM on $\mathbb{Z}^d$ exist and can be written explicitly as analytic functions of $1 - p$.

One of the main motivations of the present paper was to shade some light on obscurity of the regime $q < 1$ of the RCM in $\mathbb{Z}^d$, whose investigation has been neglected in the literature mainly due to the fact that the Potts models are at the values $q = 1, 2, \ldots$, but also due to the lack of validity of the FKG inequality (see comments at the beginning of Section 3.6 in ref. 5). The methods based on analyticity bypass obstacles due to troubles of probabilistic origin and indeed, at least in the region of convergence of the expansion, gives very detailed information about the relevant quantities, which may be used to catch also some general aspects of the behavior of the system in the regime $q < 1$. The analyticity is a new result also in the regime $q \geq 1$, but here of course its utility is more questionable, given the huge amount of information about the model one can get via probabilistic methods based on FKG inequalities.

Taking advantages on the robustness and flexibility of cluster expansion techniques, the analysis of analyticity of RCM could also be extended to a class of graph much more general than the regular lattices like $\mathbb{Z}^d$. We plan to do this in a future paper.

The p near zero expansion is an "high temperature" expansion very similar to the one developed recently for the antiferromagnetic Potts model on graphs,[16,14] while the p near 1 expansion is a "low temperature" expansion, generalizing (to $q \neq 1$) the expansion developed in ref. 2 and 15 for percolation in $\mathbb{Z}^d$. Another low temperature contour expansion for RCM has been previously developed in ref. 11 (and references therein), however for a quite different situation respect to the one we consider here. In ref. 11, authors works very close to the critical point (taking the parameter q very large to raise the temperature), while we are very far from it.

From analyticity also follows immediately that connectivities decay exponentially and we can estimate the inverse correlation length in both regimes p small and p large. For $| p |< \varepsilon_q$ we find that the inverse correlation length behaves as $| \ln(p/q) | + O(1)$. For $| 1 - p |< \delta_q$ the inverse correlation length (of the finite connectivity) behaves as $2(d-1) | \ln(1-p) | + O(1)$.

Analyticity results above also allow to extend the Theorem 4.2. in ref. 1 for values of q in the interval $0 < q < 1$. Namely, Theorems 3.1 and 4.4 below immediately imply that for the RCM on $\mathbb{Z}^d$, it exists a critical value $0 < p_c(q) < 1$ such that for $p < p_c(q)$ the probability to have a infinite open cluster is zero, while for $p > p_c(q)$ is one, and this is true for any $q > 0$.

The paper is organized as follows. In Section 2 we give some definitions and we introduce the model. In Section 3 we study the highly subcritical phase. At the beginning of this section we resume our results concerning the phase p sufficiently small (Theorem 3.1), The rest of the section is devoted to the proof of this theorem. More specifically, in Section 3.2. we construct a polymer expansion of connectivity functions and the pressure in the supercritical regime. In Section 3.3. we prove that this expansion is absolutely convergent for p sufficiently small. In Section 4 we perform the analysis of the supercritical phase. Namely, in Section 4.1. we give some more definitions and properties about cut sets in $\mathbb{Z}^d$. At the end of the subsection, we state the results on the supercritical phase (Theorem 4.4). In

Section 4.2 we construct the polymer expansion for the connectivity functions. In Section 4.3 we show that this expansion is absolutely convergent for p sufficiently near 1.

2. THE MODEL

For any finite or countable set V, we will denote by $|V|$ the cardinality of V. We denote by $\mathrm{P}_2(V)$ the set of all subsets $U \subset V$ such that $|U| = 2$. A *graph* is a pair $G = (V, E)$ with V being a countable set, and $E \subset \mathrm{P}_2(V)$. The elements of V are called *vertices* of G and the elements of E are called *edges* of G. A graph $G = (V, E)$ is *finite* if $|V| < \infty$, and infinite otherwise. If $V' \subseteq V$ and $E' \subseteq E$, then G' is a *subgraph* of G, written as $G' \subseteq G$. A graph $G = (V, E)$ is *connected* if for any pair B, C of subsets of V such that $B \cup C = V$ and $B \cap C = \emptyset$, there is an edge $e \in E$ such that $e \cap B \neq \emptyset$ and $e \cap C \neq \emptyset$. A graph $G = (V, E)$ is a called a *tree* if it is connected and $|E| = |V| - 1$. Given $G = (V, E)$ and $R \subset V$, let $E|_R = \{\{x, y\} \in E : x \in R, y \in R\}$. Then the graph $G|_R = (R, E|_R)$ is called the *the restriction of G to R*. We say that $R \subset V$ is *connected* if $G|_R$ is connected. Analogously, Given $G = (V, E)$ connected and $\eta \subset E$, let $V|_\eta = \{x \in V : x \in e$ for some $e \in \eta\}$. We call $V|_\eta$ the *support* of η. We say that a edge set $\eta \in E$ is *connected* if the graph $g = (V|_\eta, \eta)$ is connected.

We regard the d-dimensional cubic lattice as the infinite graph $\mathbb{L}^d = (\mathbb{V}, \mathbb{E})$ with vertex set $\mathbb{V} = \mathbb{Z}^d = \{x = (x_1, \ldots, x_d) : \ x_i \in \mathbb{Z}\}$ and edge set formed by all the nearest neighbor pairs, i.e., $\mathbb{E} = \{e = \{x, y\} \subset \mathbb{V} : |x - y| = 1\}$ where $|x - y| = \sum_{i=1}^d |x_i - y_i|$ is the graph distance.

For any non empty $R \subset \mathbb{V}$, the set $\partial_e R = \{e \in \mathbb{E} : |e \cap R| = 1\}$ is called the *(edge) boundary* of R. The set $\partial_v^{\mathrm{int}} R = \{x \in R : |\{x\} \cap \partial_e R| = 1\}$ is called the *internal vertex boundary* of R. Given $X \subset \mathbb{V}$, the minimal tree distance of X is defined as $d^{\mathrm{tree}}(X) = \min_{\tau \in \mathcal{T}_X} \sum_{\{x,y\} \in \tau} |x - y|$ where $\mathcal{T}_X$ is the set of trees with vertex set X. Note that $d^{\mathrm{tree}}(X) = |X| - 1$ if X is connected. We will denote by Λ_N the square box of size $2N + 1$ in $\mathbb{Z}^d$ centered at the origin.

We define initially the model on a finite $V \subset \mathbb{Z}^d$. Let $G = \mathbb{L}^d|_V = (V, E)$ with $E = \mathbb{E}|_V$. For each edge $e \in E$ we define a binary random variable $\omega(e)$, which can assume the values $\omega(e) = 1$ (open edge) and $\omega(e) = 0$ (closed edge). A configuration ω of the process is a function $\omega : E \to \{0, 1\} : e \mapsto \omega(e)$. We call Ω_V the configuration space, i.e. the set of all possible configurations of random variables $\omega(e)$ at the edges $e \in E$ of the graph G. Given $\omega \in \Omega_V$ we denote by $O(\omega)$ the subset of E given by $O(\omega) = \{e \in E : \omega(e) = 1\}$ and by $C(\omega)$ the set $C(\omega) = \{e \in E : \omega(e) = 0\}$. An *open connected component* g of ω is a connected subgraph $g = (V_g, E_g)$ of G such that $E_g \neq \emptyset$, $\omega(e) = 1$ for all $e \in E_g$, and $\omega(e) = 0$ for all $e \in \partial g$, where $\partial g = \{e \in \mathbb{E} - E_g : \ e \cap V_g \neq \emptyset\}$. A vertex $x \in V$ such that $\omega(e) = 0$ for all e adjacent to x is an *isolated vertex* of ω. The probability

$P_V(\omega)$ to see the system in the configuration $\omega \in \Omega_V$ is defined as

$$P_V(\omega) = \frac{1}{Z_V(p,q)} p^{|O(\omega)|}(1-p)^{|C(\omega)|} q^{k(\omega)}$$

where $p \in (0, 1)$, $q \in (0, \infty)$, and $k(\omega)$ is the number of connected open components of the configuration ω plus the number of isolated vertices; the normalization constant $Z_V(p, q)$, usually called the partition function of the system, is given by

$$Z_V(p,q) = \sum_{\omega \in \Omega_V} p^{|O(\omega)|}(1-p)^{|C(\omega)|} q^{k(\omega)} \tag{2.1}$$

The "pressure" of the system is defined as the following function

$$\pi_V(p,q) = \frac{1}{|V|} \ln Z_V(p,q)$$

In order to define the RCM on $\mathbb{L}^d$, we will need to introduce the concept of boundary condition. Let Ω be the set of all configurations in $\mathbb{L}^d$, i.e. the set of all functions ω such that $\omega : \mathbb{E} \to \{0, 1\}$. Let $V \subset \mathbb{V}$ a *finite* set and let $\mathbb{L}^d \mid_V$ be the restriction of $\mathbb{L}^d$ to V. Given now $\xi \in \Omega$, let Ω_V^ξ the (finite) subset of Ω of all configurations $\omega \in \Omega$ such that $\omega(e) = \xi(e)$ for $e \notin \mathbb{E} \mid_V$. For $\omega \in \Omega_V^\xi$, let us also denote by ω_V the restriction of ω on $\mathbb{E} \mid_V$. Note that ω_V does not depend on ξ. We now denote P_V^ξ the random cluster probability measure in Ω_V^ξ on the finite subgraph $\mathbb{L}^d \mid_V$ of the d-dimensional cubic lattice $\mathbb{L}^d$ with boundary conditions ξ as

$$P_V^\xi(\omega) = \frac{1}{Z_V^\xi(p,q)} p^{|O(\omega_V)|}(1-p)^{|C(\omega_V)|} q^{k_V^\xi(\omega)} \tag{2.2}$$

where $Z_V^\xi(p, q)$ is the partition function given by

$$Z_V^\xi(p,q) = \sum_{\omega \in \Omega_V^\xi} p^{|O(\omega_V)|}(1-p)^{|C(\omega_V)|} q^{k_V^\xi(\omega)} \tag{2.3}$$

and $k_V^\xi(\omega)$ is the number of *finite* connected open component (open clusters) of the configuration ω (which agrees with ξ outside V) which intersect V plus the number of isolated vertices in V. Note that $k_V^\xi(\omega)$ is the only term in (2.2) and (2.3) depending on boundary conditions ξ.

Two extremal boundary conditions play a central role, namely the *free boundary* condition, in which $\xi(e) = 0$ for all $e \in \mathbb{E}$ and the *wired boundary condition*, in which $\xi(e) = 1$ for all $e \in \mathbb{E}$. According to the definition above, for a fixed configuration ω with $\xi = 0$ outside V the number $k^0(\omega)$ is actually the number of open components in the finite sub graph $\mathbb{L}^d \mid_V$ plus the isolated vertices in V, while if $\xi = 1$ outside V, all open components in $\mathbb{L}^d \mid_V$ which touch the boundary has not to be counted computing the number $k^1(\omega)$, since they belong to the infinite open cluster. Thus $k^1(\omega)$ is actually the number of finite open connected component in ω which does not touch the boundary plus isolated vertices which does not belong to the boundary.

It is important to remark here that in the above definition of $k_V^\xi(\omega)$ we compute only the *finite* connected components because we are adopting the so called "infinity-wired boundary condition" convention, see e.g. Definition 2.1 in ref. 10 or Section 2.3 in ref. 9. By this convention, all infinite open clusters eventually intersecting V are counted as one, i.e., as if all these clusters were connected at infinity (wired at infinity). In the literature one can also find the so-called "infinity-free boundary condition" convention, in which all open clusters, whether finite or infinite, are counted in the number $k(\omega)$. In this case all infinite clusters intersecting V are regarded as separate. This is e.g. the convention adopted in the survey.[5] In the rest of the paper we will only consider the free ($\xi = 0$) and wired ($\xi = 1$) boundary conditions, for which "infinity-free convention" and "infinity-wired convention" are equivalent and we adopted the latter only because leads to simpler definitions.

Definition 2.1. *Let $\Lambda_N = \{x \in \mathbb{Z}^d : |x| \leq N\} \subset \mathbb{V}$ be the square box of size $2N+1$ at the origin. Let ξ be a boundary condition. Then the pressure of the random cluster model with parameters q and p and boundary condition ξ on $\mathbb{L}^d$ is*

$$\pi(p,q) = \lim_{N\to\infty} \frac{1}{|\Lambda_N|} \ln Z^\xi_{\Lambda_N}(q) \tag{2.4}$$

In Definition 2.1, instead of choosing a fixed boundary condition ξ, one can also think to allow a whole sequence ξ_N of boundary conditions, one for each $\Lambda_N \in \mathbb{V}$. However, as shown in ref. 6 (see also ref. 5), this adds no extra generality.

Remark 2.2. *It is easy to prove that this limit is independent on the boundary condition. As a matter of fact, let $\xi, \omega \in \Omega_{\mathbb{G}}$ and define ω_N^ξ by*

$$\omega_N^\xi(e) = \begin{cases} \omega(e) & \text{if } e \in \mathbb{E}\,|_{\Lambda_N} \\ \xi(e) & \textit{otherwise} \end{cases}$$

Then, for all ξ

$$k^1_{\Lambda_N}(\omega^1_N) \leq k^\xi_{\Lambda_N}(\omega^\xi_N) \leq k^0_{\Lambda_N}(\omega^0_N) \leq k^1_{\Lambda_N}(\omega^1_N) + |\partial\Lambda_N|$$

whence

$$Z^1_{\Lambda_N}(p,q) \leq Z^\xi_{\Lambda_N}(p,q) \leq Z^0_{\Lambda_N}(p,q) \leq Z^1_{\Lambda_N}(p,q)q^{|\partial\Lambda_N|}, \quad \text{if } q \geq 1$$

while for $q < 1$ we have simply to reverse all inequalities above. Now taking the logarithms, dividing by $|\Lambda_N|$ and using that $|\partial\Lambda_N| / |\Lambda_N| \to 0$ as $N \to \infty$ we are done.

In order to define the connectivity functions, we give some preliminary definitions. An *animal* in $\mathbb{L}^d$ is a connected subgraph $g = (V_g, E_g)$ of $\mathbb{L}^d$ with vertex set V_g finite and edge set E_g non empty. We'll denote by $\mathcal{A}$ the set of animals in $\mathbb{L}^d$ and by $\mathcal{A}_N$ the set of animal in the box Λ_N. We say that two animals $g_1 = (V_{g_1}, E_{g_1})$ and $g_2 = (V_{g_2}, E_{g_2})$ in $\mathbb{L}^d$ are *compatible* and we write $g_1 \sim g_2$ if $V_{g_1} \cap V_{g_2} = \emptyset$ (hence consequently $E_{g_1} \cap E_{g_2} = \emptyset$). Otherwise we say that g_1 and g_2 are incompatible and write $g_1 \not\sim g_2$.

We are now ready to give the definition of connectivity functions.

Definition 2.3. *Let $X \subset \mathbb{Z}^d$ finite. Let $\{\Lambda_N\}_{N\in\mathbb{N}}$ be a sequence of square boxes. Let ξ be a boundary condition. Then we define, if it exists, the connectivity function of the set X of the random cluster model with parameters q and p and boundary condition ξ on $\mathbb{L}^d$ as*

$$\phi_{p,q,\xi}(X) = \lim_{N\to\infty} \sum_{\substack{\omega\in\Omega^{\xi}_{V_N}:\ \exists g\in\mathcal{A}_N:\\ E_g\in O(\omega),\quad X\subset V_g}} P^{\xi}_{\Lambda_N}(\omega) \tag{2.5}$$

The truncated connectivity function of the set X of the random cluster model with parameters q and p and boundary condition ξ is defined as

$$\phi^{\mathrm{f}}_{p,q,\xi}(X) = \lim_{N\to\infty} \sum_{\substack{\omega\in\Omega^{\xi}_{\mathbb{G}_N}:\ \exists g\in\mathcal{A}_N:\ E_g\in O(\omega)\\ X\subset V_g,\ V_g\cap\ \partial^{\mathrm{int}}_v\Lambda_N=\emptyset}} P^{\xi}_{\Lambda_N}(\omega) \tag{2.6}$$

We recall that $\phi^{\mathrm{f}}_{p,q,\xi}(X)$ coincides with the *connectivity function* in the subcritical phase (since in that case there is no infinite open cluster in the system).

For $q > 1$ is immediate to check that, for any boundary condition ξ

$$\phi_{p,q,0}(X) \le \phi_{p,q,\xi}(X) \le \phi_{p,q,1}(X) \tag{2.7}$$

$$\phi^{\mathrm{f}}_{p,q,0}(X) \le \phi^{\mathrm{f}}_{p,q,\xi}(X) \le \phi^{\mathrm{f}}_{p,q,1}(X) \tag{2.8}$$

Hence if one is able to prove e.g. that $\phi^{\mathrm{f}}_{p,q,1}(X) = \phi^{\mathrm{f}}_{p,q,0}(X)$ then he has also proven for free that $\phi^{\mathrm{f}}_{p,q,1}(X) = \phi^{\mathrm{f}}_{p,q,\xi}(X) = \phi^{\mathrm{f}}_{p,q,0}(X)$ for any fixed boundary condition ξ, *as far as* $q \ge 1$.

For $q < 1$ we cannot arrive to the same conclusion, since (2.7) and (2.8) are false when $q < 1$.

As it will be shown below we are able to prove using cluster expansion techniques for all $q > 0$ that $\phi_{p,q,1}(X) = \phi_{p,q,0}(X)$ for p sufficiently small and that $\phi^{\mathrm{f}}_{p,q,1}(X) = \phi^{\mathrm{f}}_{p,q,0}(X)$ for p sufficiently near 1. This result can be generalized, at least in the subcritical phase, to a wider class of boundary conditions. For example it is very easy to cover the case of boundary conditions such that the cardinality of each set of vertices in the boundary connected through the boundary conditions itself is uniformly bounded. It is unclear for us if it is possible to further generalize our expansions in order to include all boundary conditions in the whole regime $q > 0$.

However, by preliminary calculations this would increase complexity of notations and definitions used in our expansion, so we preferred to focus our attention to the cases $\xi = 0, 1$, which are the more interesting. Hence hereafter we treat only free and wired boundary conditions.

Note finally that the percolation probability $\theta^{\xi}_{p,q}(0 \leftrightarrow \infty)$, i.e. the probability that there is an open cluster passing through the origin 0 is defined in term of connectivity functions as

$$\theta^{\xi}_{p,q}(0 \leftrightarrow \infty) = 1 - \phi^{\mathrm{f}}_{p,q,\xi}(X = \{0\}) \tag{2.9}$$

The critical percolation probability $p^{\xi}_{c}(q)$ at a fixed value of q for $\mathbb{Z}^d$ is the value of p defined by

$$p^{\xi}_{c}(q) = \sup_{p \in [0,1]} \left\{ p : \theta^{\xi}_{p,q}(0 \leftrightarrow \infty) = 0 \right\} \tag{2.10}$$

3. THE SUBCRITICAL PHASE

3.1. Results in the Subcritical Phase

We begin this section stating our results in the subcritical regime.

Theorem 3.1. *For any $q > 0$, let p so small that $| p/(1-p) | \le r_q$ with*

$$r_q = \min \left\{ \frac{q}{24d^2}, \quad \frac{1}{24d^2} \right\} \tag{3.1}$$

Then:

(a) *The pressure of RCM on $\mathbb{Z}^d$, defined in (2.4) is analytic as a function of p.*
(b) *The infinite volume connectivity functions $\phi^{\xi}_{p,q}(X)$ with $\xi = 0, 1$ of the RCM on $\mathbb{Z}^d$ defined in the limit (2.5) exist, are both equal to a function $\phi_{p,q}(X)$ which is analytic as a function of p.*

Moreover $| \phi_{p,q}(X) |$ admit the upper bound

$$| \phi_{p,q}(X) | \le C_d \left[\frac{4d^2}{q} \frac{p}{1-p} \right]^{d^{\mathrm{tree}}(X)} \tag{3.2}$$

where $d^{\mathrm{tree}}(X)$ is the tree distance of X in $\mathbb{L}^d$ and C_d is a constant.

3.2. Polymer Expansion in the Subcritical Regime

We will use the shorter notations $\mathbb{E}_N = \mathbb{E}\mid_{\Lambda_N}$, $k^{\xi}_{\Lambda_N} = k^{\xi}_N$, $\omega_{\Lambda_N} = \omega_N$, $\Omega_N = \Omega_{\Lambda_N}$.

Fix a $X \subset \Lambda_N - \partial_v^{\text{int}}\Lambda_N$ (X does not touch the boundary). The finite volume free and wired connectivity function can be rewritten as

$$\phi^N_{p,q,\xi=0,1}(X) = \frac{1}{\tilde{Z}^{\xi}_N(p,q)} \sum_{\substack{\omega\in\Omega^{\xi}_{V_N}:\ \exists g\in\mathcal{A}_N: \\ E_g\subset O(\omega),\quad X\subset V_g}} \lambda^{|O(\omega_N)|} q^{k^{\xi}_N(\omega)} \tag{3.3}$$

where

$$\tilde{Z}^{\xi}_N(p,q) = \sum_{\omega\in\Omega^{\xi}_{\Lambda_N}} \lambda^{|O(\omega_N)|} q^{k^{\xi}_N(\omega)} = (1-p)^{|\mathbb{E}_N|} Z^{\xi}_N(p,q) \tag{3.4}$$

and

$$\lambda = \frac{p}{1-p} \tag{3.5}$$

We recall that $k^0_N(\omega)$ is the number of open components of ω_N plus isolated vertices, while $k^1_N(\omega)$ is the number of open connected component in ω_N which do not intersect the boundary plus isolated vertices which does belong to the boundary $\partial_v^{\text{int}}\Lambda_N$.

A configuration $\omega \in \Omega^{\xi}_N$ is completely specified by the set of open edges $O(\omega_N)$ in $\mathbb{E}_N$. Let now $\{E_1, \ldots, E_n\}$ be the connected components of $O(\omega_N)$. To each E_i we can associate an animal $g_i \in \mathcal{A}$ such that $V_{g_i} = \mathbb{V}\mid_{E_i}$, $E_{g_i} = E_i$. Then to each $\omega \in \Omega^{\xi}_N$ can be associated a (unordered) set of animals $\{g_1, \ldots, g_n\}_{\omega_N} \subset \mathcal{A}_N$ such that $\cup_{i=1}^n E_{g_i} = O(\omega_N)$ and for all $i, j \in \mathrm{I}_n$, $g_i \sim g_j$. Observe that this one to one correspondence $\omega_N \leftrightarrow \{g_1, \ldots, g_n\}$ yields

$$|\, O(\omega_N)| = \sum_{i=1}^{n} |\, E_{g_i}| \tag{3.6}$$

$$\sum_{\omega\in\Omega^{\xi}_N} (\cdot) = \sum_{n\geq 0} \sum_{\substack{\{g_1,\ldots,g_n\}\subset\mathcal{A}_N \\ g_i\sim g_j}} (\cdot) \qquad \sum_{\substack{\omega\in\Omega^{\xi}_{\mathbb{G}_N}:\ \exists g\in\mathcal{A}_N: \\ E_g\subset O(\omega),\quad X\subset V_g}} (\cdot) = \sum_{n\geq 1} \sum_{\substack{\{g_1,\ldots,g_n\}\subset\mathcal{A}_N \\ g_i\sim g_j,\ X\subset V_{g_1}}} (\cdot) \tag{3.7}$$

where for $n = 0$ the unordered n-uple $\{g_1, \ldots, g_n\}$ is the empty set.

We will now rewrite the partition function (3.4) and the connectivity function (3.3) in terms of the animals introduced above. We start by considering the case $\xi = 0$. Let us denote by $V^{\text{iso}}_{\omega_N}$ the subset of Λ_N formed by the isolated vertices in the configuration ω_N, and let $\{g_1, \ldots, g_n\}_{\omega_N}$ be the animals uniquely associated to $O(\omega_N)$. Then, by definition,

$$k_N^0(\omega) = n + |V_{\omega_N}^{\text{iso}}|$$

and since

$$|V_{\omega_N}^{\text{iso}}| = |\Lambda_N| - \sum_{i=1}^{n} |V_{g_i}|$$

we obtain

$$k_N^0(\omega) = |\Lambda_N| - \sum_{i=1}^{n} [|V_{g_i}| - 1] \tag{3.8}$$

Using now (3.6), (3.7) and (3.8), the partition function $\tilde{Z}_{V_N}^0(p,q)$ defined in (3.4) can be rewritten as

$$\tilde{Z}_N^0(p,q) = q^{|\Lambda_N|}\,\Xi_N^0(p,q) \tag{3.9}$$

where

$$\Xi_N^0(p,q) = 1 + \sum_{n\geq 1} \sum_{\substack{\{g_1,\ldots,g_n\}\subset\mathcal{A}_N \\ g_i \sim g_j}} \prod_{i=1}^{n} \frac{1}{q^{|V_{g_i}|-1}} \lambda^{|E_{g_i}|} \tag{3.10}$$

and

$$\phi_{p,q,\xi=0}^N(X) = \frac{1}{\Xi_{\Lambda_N}^0(p,q)} \sum_{n\geq 1} \sum_{\substack{\{g_1,\ldots,g_n\}\subset\mathcal{A}_N \\ g_i \sim g_j,\ X\subset V_{g_1}}} \prod_{i=1}^{n} \frac{1}{q^{|V_{g_i}|-1}} \lambda^{|E_{g_i}|} \tag{3.11}$$

The case $\xi = 1$ is slightly more involved. We first find an expression of $k_N^1(\omega)$ in terms of the animals $\{g_1,\ldots g_n\}$ uniquely associated to $O(\omega_N)$. The set $I_n = \{1,2,\ldots,n\}$ is naturally partitioned in the disjoint union of two sets $\mathrm{I}_n^{\text{int}}$ and I_n^{∂} defined as

$$\mathrm{I}_n^{\text{int}} = \{i \in \mathrm{I}_n : V_{g_i} \cap \partial_v^{\text{int}}\Lambda_N = \emptyset\}; \qquad \mathrm{I}_n^{\partial} = \left\{i \in \mathrm{I}_n : V_{g_i} \cap \partial_v^{\text{int}}\Lambda_N \neq \emptyset\right\}$$

Denoting now shortly $\Lambda_N - \partial_v^{\text{int}}\Lambda_N = V_N^{\text{int}}$ and, for $i \in \mathrm{I}_n^{\partial}$, $V_{g_i}^{\text{int}} = V_{g_i} - \partial_v^{\text{int}}\Lambda_N$, we have

$$k_N^1(\omega) = |V_N^{\text{int}}| - \sum_{i\in\mathrm{I}_n^{\text{int}}} (|V_{g_i}| - 1) - \sum_{i\in\mathrm{I}_n^{\partial}} |V_{g_i}^{\text{int}}| \tag{3.12}$$

Hence in the case $\xi = 1$ we get

$$\tilde{Z}_N^1(p,q) = q^{|V_N^{\text{int}}|}\,\Xi_N^1(p,q)$$

where

$$\Xi^1_N(p,q) = 1 + \sum_{n\geq 1} \sum_{\substack{\{g_1,\ldots,g_n\}\subset\mathcal{A}_N \\ g_i\sim g_j}} \prod_{i\in \mathrm{I}_n^{\mathrm{int}}} \frac{1}{q^{|V_{g_i}|-1}} \lambda^{|E_{g_i}|} \prod_{i\in \mathrm{I}_n^{\partial}} \frac{1}{q^{|V_{g_i}^{\mathrm{int}}|}} \lambda^{|E_{g_i}|} \tag{3.13}$$

and

$$\phi^N_{p,q,\xi=1}(X) = \frac{1}{\Xi^1_{\mathbb{G}|_N}(p,q)} \sum_{n\geq 1} \sum_{\substack{\{g_1,\ldots,g_n\}\subset\mathcal{A}_N \\ g_i\sim g_j,\ X\subset V_{g_1}}} \prod_{i\in \mathrm{I}_n^{\mathrm{int}}} \frac{1}{q^{|V_{g_i}|-1}} \lambda^{|E_{g_i}|} \prod_{i\in \mathrm{I}_n^{\partial}} \frac{1}{q^{|V_{g_i}^{\mathrm{int}}|}} \lambda^{|E_{g_i}|} \tag{3.14}$$

Given $g = (V_g, E_g) \in \mathcal{A}$ we define now the activity of g as

$$\rho(g) = q^{-(|V_g|-1)} \lambda^{|E_{g_i}|} \tag{3.15}$$

We also define a ξ-dependent activity as

$$\rho^\xi(g) = \begin{cases} \rho(g) & \text{if } \xi = 0 \text{ or if } \xi = 1 \text{ and } V_g \cap \partial_v^{\mathrm{int}} \Lambda_N = \emptyset \\ q^{-|V_{g_i}^{\mathrm{int}}|} \lambda^{|E_{g_i}|} & \text{if } \xi = 1 \text{ and } V_g \cap \partial_v^{\mathrm{int}} \Lambda_N \neq \emptyset \end{cases} \tag{3.16}$$

Note that $\rho^0(g)$ is the restriction of $\rho(g)$ to $\mathbb{E}_N$ and when $q < 1$ we have, for all $g \in \mathcal{A}$, that

$$|\, \rho^\xi(g) \,| \leq |\, \rho(g) \,| \qquad \text{whenever } \ q < 1 \tag{3.17}$$

We will use the shorthand notations

$$\mathbf{g}_n \equiv (g_1, \ldots, g_n)\,; \qquad \rho^\xi(\mathbf{g}_n) \equiv \rho^\xi(g_1)\cdots\rho^\xi(g_n) \qquad \rho(\mathbf{g}_n) \equiv \rho(g_1)\cdots\rho(g_n)$$

Define further the hard core pair potential between two subsets R_i, R_j as

$$U(g_i, g_j) = \begin{cases} +\infty & \text{if } g_i \not\sim g_j \\ 0 & \text{otherwise,} \end{cases} \tag{3.18}$$

and denote shortly

$$U(\mathbf{g}_n) = \sum_{1\leq i<j\leq n} U(g_i, g_j)$$

Then for $\xi = 0, 1$ we collect (3.10), (3.11), (3.13) and (3.14) as

$$\phi^N_{p,q,\xi}(X) = \frac{1}{\Xi^\xi_{\Lambda_N}(p,q)} \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{\mathbf{g}_n\in\mathcal{A}_N^n \\ \exists i\in \mathrm{I}_n:\, g_i\supset X}} \rho^\xi(\mathbf{g}_n) e^{-U(\mathbf{g}_n)} \tag{3.19}$$

and

$$\Xi^\xi_N(p,q) = \left[1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\mathbf{g}_n\in\mathcal{A}_N^n} \rho^\xi(\mathbf{g}_n) e^{-U(\mathbf{g}_n)} \right] \tag{3.20}$$

where $I_n = \{1, 2, \cdots, n\}$ and $\mathcal{A}_N^n$ is the n-times cartesian product of $\mathcal{A}_N$, i.e. elements of $\mathcal{A}^n$ are ordered n-ples of elements of $\mathcal{A}_N$. The factor 1 in r.h.s. of (3.20) is the contribution of the configuration in which all edges in Λ_N are closed. Observe that the partition function is rewritten as a hard core polymer gas partition function in which the polymers are animals g of $\mathbb{Z}^d$ with activity $\rho^\xi(g)$.

It is now easy to rewrite this ratio (between two finite sums) as an infinite series. In order to do that we define a new activity depending of a real parameter α as

$$\rho_\alpha^\xi(g) = \begin{cases} (1+\alpha)\rho^\xi(g) & \text{if } X \subset V_g \\ \rho^\xi(g) & \text{otherwise} \end{cases} \tag{3.21}$$

and a new α-depending partition function

$$\Xi_{N,\alpha}^\xi(p,q) = \left[1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\mathbf{g}_n \in \mathcal{A}_N^n} \rho_\alpha^\xi(\mathbf{g}_n) e^{-U(\mathbf{g}_n)}\right] \tag{3.22}$$

where, of course $\rho_\alpha^\xi(\mathbf{g}_n) = \rho_\alpha^\xi(g_1) \cdots \rho_\alpha^\xi(g_n)$.

So, by construction

$$\phi_{p,q,\xi}^N(X) = \frac{d}{d\alpha} \ln \Xi_{N,\alpha}^\xi(p,q)\bigg|_{\alpha=0}$$

Now, by standard cluster expansion it is well known that

$$\ln \Xi_{N,\alpha}^\xi(p,q) = \sum_{n\geq 1} \frac{1}{n!} \sum_{\mathbf{g}_n \in \mathcal{A}_N^n} \rho_\alpha^\xi(\mathbf{g}_n) \Phi^T(\mathbf{g}_n) \tag{3.23}$$

where the Ursell coefficients $\Phi^T(\mathbf{g}_n)$ are given by

$$\Phi^T(\mathbf{g}_n) = \begin{cases} \displaystyle\sum_{\substack{E \in P_2(I_n) \\ (I_n, E) \in \mathcal{G}_n}} \prod_{\{i,j\} \in E} (e^{-U(g_i,g_j)} - 1) & \text{if } n \geq 2 \\ 1 & \text{if } n = 1. \end{cases} \tag{3.24}$$

where $\mathcal{G}_n$ denotes the set of all connected graphs with vertex set I_n.

Deriving now the series in r.h.s. of (3.23) term by term with respect to α and evaluating the result at $\alpha = 0$, it is clear, see (3.21), that the only non vanishing terms are those associated to configurations $\mathbf{g}_n$ in which at least one among the R_i's is such that $X \subset g_i$. Thus we obtain

$$\phi_{p,q,\xi}^N(X) = \sum_{n=1}^\infty \frac{1}{n!} \sum_{\substack{\mathbf{g}_n \in \mathcal{A}_N^n \\ \exists i \in I_n:\ X \subset V_{g_i}}} k(\mathbf{g}_n) \Phi^T(\mathbf{g}_n) \rho^\xi(\mathbf{g}_n) \tag{3.25}$$

where $k(\mathbf{g}_n) =| \{i \in I_n : X \subset V g_i\} |$. Note that $k(\mathbf{g}_n) \leq n$.

We also define a function on the whole cubic lattice $\mathbb{L}^d$ (hence not depending on boundary conditions) as follows

$$\phi_{p,q}(X) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{\mathbf{g}_n \in \mathcal{A}^n \\ \exists i \in \mathrm{I}_n :\ X \subset V_{g_i}}} k(\mathbf{g}_n) \Phi^T(\mathbf{g}_n) \rho(\mathbf{g}_n) \tag{3.26}$$

By the standard theory of polymer expansion the series above are absolutely convergent series when the polymer activity is sufficiently small, accordingly to the so called Kotecky-Preiss Condition[(8)] (see also ref. 13), namely

$$\sum_{n \geq n_0} f_n(\rho) e^{an} \leq a \tag{3.27}$$

where in this case $n_0 = \min_{g \in \mathcal{A}} |V_g| = 2$ and

$$f_n(\rho) = \sup_{x \in \mathbb{V}} \sum_{\substack{g \in \mathcal{A}: \\ x \in V_g,\ |V_g| = n}} |\rho(g)| \qquad \text{and} \qquad f_n(\rho^\xi) = \sup_{x \in \Lambda_N} \sum_{\substack{g \in \mathcal{A}_N \\ x \in V_g,\ |V_g| = n}} |\rho^\xi(g)$$

3.3. Proofs of Theorem 3.1

Recalling now definitions (3.15) and (3.16), and using the rough bounds $\sum_{g \in \mathcal{A}:\ x \in V_g,\ |V_g| = n} \leq (2d)^{2(n-1)}$ we get immediately

$$f_n(\rho) \leq (\varepsilon_*)^{n-1} \leq \varepsilon^{n-1} \quad \text{and} \quad f_n(\rho^\xi) \leq \varepsilon^{n-1} \tag{3.28}$$

where

$$\varepsilon_* = \frac{4d^2\lambda}{q} \quad \text{and} \quad \varepsilon = \max\left\{\frac{4d^2\lambda}{q}, 4d^2\lambda\right\} \tag{3.29}$$

Thus, inserting (3.28) into (3.27) the Kotecky-Preiss condition is satisfied e.g. choosing $a = \ln 2$, by $\varepsilon < 1/6$. Hence recalling definition (3.29), we obtain that the series (3.25) and (3.26) are absolutely convergent at least for all λ in the disk $|\lambda| \leq r_q$ with

$$r_q = \min\left\{\frac{q}{24d^2}, \quad \frac{1}{24d^2}\right\} \tag{3.30}$$

Finally we prove the following theorem.

Theorem 3.2. *For any fixed $q > 0$, $\xi = 0, 1$ and λ in the disk $|\lambda| < r_q$*

$$\lim_{N \to \infty} \phi^N_{p,q,\xi}(X) = \phi_{p,q}(X)$$

where $\phi_{p,q}(X)$ is the function defined in (3.26).

Proof: Let us consider the case $\xi = 1$, since the case $\xi = 0$ is easier because $\rho^0(g) = \rho(g)$.

$$
\begin{aligned}
&|\phi_{p,q}(X) - \phi^N_{p,q,\xi=1}(X)| \\
&\leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{\mathbf{g}_n \in \mathcal{A}^n \\ \exists k:\ X \subset V_{g_k} \\ \exists j:\ g_j \not\subset \Lambda_N}} k(\mathbf{g}_n) \mid \Phi^T(\mathbf{g}_n) \mid\mid \rho(\mathbf{g}_n) \mid \\
&+ \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{\mathbf{g}_n \in \mathcal{A}^n_N \\ \exists k:\ X \subset V_{g_k} \\ \exists j:\ V_{g_j} \cap \partial^{\mathrm{int}}_v \Lambda_N \neq \emptyset}} k(\mathbf{g}_n) \mid \Phi^T(\mathbf{g}_n) \mid\mid \rho^1(\mathbf{g}_n) - \rho(\mathbf{g}_n) \mid
\end{aligned}
$$

Now, the first term of the r.h.s. of this inequality is, for $\varepsilon < 1/6$, a convergent series clearly at least of the order $(4d^2\lambda/q)^{d(X,\partial^{\mathrm{int}}_v \Lambda_N)}$ (here $d(X, \partial^{\mathrm{int}}_v \Lambda_N) = \min\{|x - y|:\ x \in X,\ y \in \partial^{\mathrm{int}}_v \Lambda_N\}$), since one among the $g_1, \ldots, g_n$ has to contain X and another has to intersect $\mathbb{V} - \Lambda_N$. Recall that the sets $g_1, \ldots, g_n$ are pairwise incompatible due to the presence of the factor $\Phi^T(\mathbf{R}_n)$.

The second term can be treated similarly, using that $| \rho^1(\mathbf{g}_n) - \rho(\mathbf{g}_n) | \leq 2 \mid \rho^1(\mathbf{g}_n) |$, and again one shows that it is of at least of the order $(4d^2\delta)^{d_{\mathbb{G}}(X,\partial^{\mathrm{int}}_v \Lambda_N)}$, with $\delta = \max\{\lambda/q, \lambda\}$. Now as $N \to \infty$ we have clearly that $d(X, \partial^{\mathrm{int}}_v \Lambda_N) \to \infty$. The proof of the case $\xi = 0$ is the same, since just the first term in the inequality above is present. $\square$

An estimate from above of the exponential decay can be obtained immediately by considering e.g.(3.19). One obtain, uniformly in N

$$
\begin{aligned}
\phi^N_{p,q,\xi}(X) &= \frac{1}{\Xi^{\xi}_{\Lambda_N}(p, q)} \sum_{n \geq 1} \frac{1}{n!} \sum_{\substack{\mathbf{g}_n \in \mathcal{A}^n_N \\ \exists i \in \mathrm{I}_n:\ g_i \supset X}} \rho^{\xi}(\mathbf{g}_n) e^{-U(\mathbf{g}_n)} \\
&\leq \sum_{g \in \mathcal{A}:\ X \subset V_g} \rho(g) \leq C_d \left(4d^2 \frac{\lambda}{q}\right)^{d^{\mathrm{tree}}(X)}
\end{aligned}
$$

for some constants C_d whenever $|\lambda| < r_q$.

To prove part a), we recall that the pressure of the random cluster model is given by (2.4). As it has been shown in the Remark 2.2, if the pressure exists, it is independent on boundary conditions. Hence we can work here with free boundary conditions $\xi = 0$ which are easier for small p. Now, by (3.4) and (3.9)

$$
\frac{1}{|\Lambda_N|} \ln Z^0_{\Lambda_N}(q) = \frac{1}{|\Lambda_N|} \ln \Xi^0_{\Lambda_N}(q) - \frac{|\mathbb{E}_N|}{\Lambda_N} \ln(1 - p) + \ln q
$$

where we recall that $\Xi_N^\xi(p, q)$ is given explicitly by equation (3.20). Hence

$$\pi(p, q) = \lim_{N\to\infty} \frac{1}{\mid \Lambda_N \mid} \ln \Xi_N^\xi(q) - B_d \ln(1 - p) + \ln q$$

Thus in order to show that the pressure exists we need to prove that the limit

$$\Pi(p, q) = \lim_{N\to\infty} \frac{1}{\mid \Lambda_N \mid} \ln \Xi_N^0(q) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathbf{g}_n \in \mathcal{A}_N^n} \Phi^T(\mathbf{g}_n)\rho(\mathbf{g}_n) \tag{3.31}$$

exists, is independent of Λ_N and has a finite radius of convergence. But this is again an immediate consequence of standard polymer expansion once condition (3.27) is verified.

4. THE SUPERCRITICAL PHASE

4.1. More Definitions and the Main Result in the Supercritical Regime

In order to study the supercritical phase we introduce the concept of minimal cut sets of $\mathbb{Z}^d$.

Definition 4.1. *A set of edges $\gamma \subset \mathbb{E}$ of $\mathbb{L}^d$ is called a cut set if the graph $\mathbb{L}^d_\gamma = (\mathbb{V}, \mathbb{E} - \gamma)$ is disconnected. A cut set $\gamma \subset \mathbb{E}$ is called a minimal cut set if for all $e \in \gamma$ the set $\gamma - e$ is not a cut set. A finite minimal cut set $\gamma \subset \mathbb{E}$ in $\mathbb{Z}^d$ will be called hereafter a fence.*

A fence $\gamma \subset \mathbb{E}$ in $\mathbb{Z}^d$ has the property (see proposition 1 in ref. 15) that the graph $\mathbb{L}^d_\gamma = (\mathbb{V}, \mathbb{E} - \gamma)$ has only two connected components $g_\gamma = (I_\gamma, E_\gamma)$ and $\gamma_c^{\text{ext}} = (O_\gamma, E_\gamma^{\text{ext}})$ the first being finite and second infinite with $E_\gamma = \mathbb{E} \mid_{I_\gamma}$ and $E_\gamma^{\text{ext}} = \mathbb{E} \mid_{O_\gamma}$. The set $I_\gamma \subset \mathbb{V}$ is called *the vertex interior of the fence γ*, and O_γ is called *the vertex exterior of the fence γ*. Analogously the set $E_\gamma \subset \mathbb{E}$ is called *the edge interior of the fence γ*, and E_γ^{ext} is called *the vertex exterior of the fence γ*.

Note that for any fence γ of $\mathbb{Z}^d$ it follows directly from the definition that $I_\gamma \cap O_\gamma = \emptyset$ and $I_\gamma \cup O_\gamma = \mathbb{V}$. Moreover $\gamma \cap E_\gamma = \gamma \cap \mathbb{E}_\gamma = E_\gamma \cap \mathbb{E}_\gamma = \emptyset$ and $E_\gamma \cup \gamma \cup \mathbb{E}_\gamma = \mathbb{E}$. From definition it also follows that $\partial_e I_\gamma = \gamma$, $E_\gamma = \mathbb{E} \mid_{I_\gamma}$ and $\mathbb{E}_\gamma = \mathbb{E} \mid_{O_\gamma}$. Moreover, any edge $e \in \gamma$ is such that $e = \{x, y\}$ with $x \in I_\gamma$ and $y \in O_\gamma$. We finally denote by $\Gamma_{\mathbb{G}}$ the set of all fences in $\mathbb{G}$.

If γ be a fence in $\mathbb{G}$ and let $x \in I_\gamma$, then it is immediate to see that for any infinite connected path of edges $E_\rho \subset \mathbb{E}$ in $\mathbb{Z}^d$ starting at x we have that $E_\rho \cap \gamma \neq \emptyset$. We'll use also the following property (see proposition 5 in ref. 15).

Proposition 4.2. *Let $a = (V_a, E_a)$ be an animal in $\mathbb{L}^d$. Then there is a unique fence γ_a such that $\gamma_a \subset (\partial_e a)^*$ and $I_{\gamma_a} \supset V_a$. Moreover, if $\gamma \subset (\partial_e a)^*$ is also a Peierls contour different from γ_a, then $I_\gamma \cap V_a = \emptyset$.*

Given a fence $\gamma \subset \mathbb{E}$ and a vertex set $X \subset \mathbb{V}$, we say that γ surrounds X and we write $\gamma \odot X$ if $X \subset I_\gamma$. We say that γ *separates* X and we write $\gamma \otimes X$, if for any animal $a = (V_a, E_a)$ such that $X \subset V_a$, $E_a \cap \gamma \neq \emptyset$, or equivalently if it happens that simultaneously $X \cap O_\gamma \neq \emptyset$ and $X \cap I_\gamma \neq \emptyset$.

Definition 4.3. *Given $X \subset \mathbb{V}$ we denote by $\gamma^X = (I_\gamma^X, E_\gamma^X)$ any fence γ with the property $X \subset I_\gamma$ and $|E_\gamma|$ minimal. We call the number $|E_\gamma|$ the fence-distance of X and denote it as $d^{\text{fen}}(X)$. Note that is $X = \{x, y\}$ then $d^{\text{fen}}(X) = 2(d-1)$ $(|x-y|+1) + 2$.*

We are now in the position to state our results concerning the supercritical regime of the Random Cluster model with free or wired boundary conditions and for p sufficiently near 1.

Theorem 4.4. *Let $(1-p)$ so small that $|(1-p)/p| < \bar{r}_q$ with*

$$\bar{r}_q = \min\left\{\frac{1}{5C_d q}, \quad \frac{1}{5C_d}\right\} \tag{4.1}$$

Then:

(a) *The pressure of RCM on $\mathbb{Z}^d$, defined in (2.4) is analytic as a function of p.*
(b) *The infinite volume connectivity functions of the RCM on $\mathbb{Z}^d$ with free and wired boundary conditions, defined in the limit (2.5), exist and are both equal to a function $\phi_{p,q}^{\text{f}}(X)$ analytic as a function of p in the region $|(1-p)/p| < \bar{r}_q$.*

   Moreover $|\phi_{p,q}^{\text{f}}(X)|$ admit the upper bound

$$|\phi_{p,q}^{\text{f}}(X)| \leq C'_d[B^d(1-p)/p]^{d^{\text{fen}}(X)}$$

where C'_d and B_d are constant depending only on d.

Remark 4.5. *The Theorem 4.4 implies that the percolation probability $\theta_{p,q}(0 \leftrightarrow \infty)$ is analytic in p and is of the order $1-(1-p)^{2d}$, since $\theta_{p,q}(0 \leftrightarrow \infty) = 1 - \phi_{p,q}^{\text{f}}(X = \{0\})$. In other words, the RCM on $\mathbb{Z}^d$, for any $q > 0$, has a percolation probability threshold strictly less than 1.*

4.2. Polymer Expansion in the Supercritical Regime

The finite volume free and wired finite connectivity functions for any $X \subset \Lambda_N - \partial_v^{\text{int}} \Lambda_N$ can be written as

$$\phi_{p,q,\xi}^{\mathrm{f},N}(X) = \frac{1}{\bar{Z}_N^{\xi}(p,q)} \sum_{\substack{\omega \in \Omega_N^{\xi}:\ \exists g \in \mathcal{A}_N:\ E_g \subset O(\omega) \\ X \subset V_g,\ V_g \cap\ \partial_v^{\text{int}} \Lambda_N = \emptyset}} \lambda^{|C(\omega_N)|} q^{k_N^{\xi}(\omega)} \tag{4.2}$$

where in this section

$$\lambda = \frac{1-p}{p}$$

and

$$\bar{Z}_N^{\xi}(p,q) = \sum_{\omega \in \Omega_N^{\xi}} \lambda^{|C(\omega_N)|} q^{k_N^{\xi}(\omega)} = p^{|\mathbb{E}_N|} Z_N^{\xi}(p,q) \tag{4.3}$$

We recall that the symbol $C(\omega_N)$ denotes the set of closed edges in $\mathbb{E}_N$ once the configuration $\omega \in \Omega_N^{\xi}$ is given.

We associate now to each edge $e \in \mathbb{E}$ a $(d-1)$-dimensional unit hypersquare $\varphi(e)$ (plaquette) which cuts perpendicularly the edge (thought immersed in $\mathbb{R}^d$) in the middle point. The vertices of the plaquette lay in the dual lattice of $\mathbb{Z}^d$. Note that the map $\varphi : \mathbb{E}^d \to \varphi(\mathbb{E}^d)$, associating to each e the corresponding $\varphi(e)$, is a one-to-one from the set $\mathbb{E}^d$ into the set $\varphi(\mathbb{E}^d)$ of plaquettes. Two edges e and e' are said *dual connected* if the corresponding plaquettes $\varphi(e)$ and $\varphi(e')$ are connected, i.e. share a $(d-2)$-dimensional side. We say that two plaquettes are *dual connected* if they share a $(d-2)$-dimensional side. In particular in $d = 3$ a plaquette turns out to be simple square and two plaquettes are connected if they share a whole unit side.

Definition 4.6. *A subset $S \subset \mathbb{E}$ is called a dual animal if it is finite and $\varphi(S)$ is a set of pairwise connected plaquettes. We say that two dual animals S and S' are compatible and we write $S \sim S'$ if $S \cup S'$ is not a dual animal (i.e. is not a set of pairwise connected plaquettes). We will denote by $\mathcal{E}$ the set of all dual animals in $\mathbb{E}$. We will also denote by $\mathcal{E}_N$ the set of dual animals in $\mathbb{E}_N$.*

Remark 4.7. *Any fence in $\mathbb{Z}^d$ is a dual animal (see proposition 2 in ref. 15).*

Definition 4.8. *Let $S \subset \mathbb{E}$ and let $\gamma \subset S$ be a fence with vertex interior I_γ and edge interior E_γ. We say that γ is minimal with respect to S if there is no other fences $\gamma' \subset S$ such that $\gamma' \cap \gamma \neq \emptyset$ and $\gamma' \subset \gamma \cup E_\gamma$. Note that a minimal fence γ can contain in its interior a fence γ' such that $\gamma \cap \gamma' = \emptyset$. Given $S \subset \mathbb{E}$ we denote by n_S the number of fences which are minimal with respect to S.*

Remark 4.9. *By the definition above and by definition 4.1, if $S \subset \mathbb{E}$ is finite, then the number of finite connected component of $(\mathbb{V}, \mathbb{E} - S)$ is exactly n_S.*

We will now give convenient expressions for $k_N^0(\omega)$ and $k_N^1(\omega)$. Let us consider first the case $k_N^1(\omega)$ which is the easier one. If we are using wired boundary conditions, then $k_N^1(\omega)$ is the number of connected components of $O(\omega_N)$ plus the isolated vertices whose support is contained in $\Lambda_N^{\rm int}$. The fences associated with any of such components is then totally contained in $\mathbb{E}_N$. This means that

$$k_N^1(\omega) = n_{C(\omega_N)} \tag{4.4}$$

Using now (4.4) the partition function $\bar{Z}_N^{\xi}(p, q)$ defined in (4.3) can be rewritten as

$$\bar{Z}_N^1(p, q) = \sum_{\omega \in \Omega_N^{\xi}} \lambda^{|C(\omega_N)|} q^{k_N^1(\omega)} = \sum_{\omega \in \Omega_N^{\xi}} \lambda^{|C(\omega_N)|} q^{n_{C(\omega_N)}} \tag{4.5}$$

and

$$\phi_{p,q,1}^{\mathrm{f},N}(X) = \frac{1}{\bar{Z}_N^1(p, q)} \sum_{\substack{\omega \in \Omega_N^{\xi}:\ \exists g \in \mathcal{A}:\ E_g \in O(\omega) \\ X \subset V_g,\ V_g \cap\ \partial_v^{\rm int} \Lambda_N = \emptyset}} \lambda^{|C(\omega_N)|} q^{n_{C(\omega_N)}}$$

The case $k_N^0(\omega)$ is more involved. Observe that the partition function

$$\bar{Z}_N^0(p, q) = \sum_{\omega \in \Omega_N^0} \lambda^{|C(\omega_N)|} q^{k_N^0(\omega)}$$

is not really a partition function, since there is no term 1. This term should correspond to the configuration in which all bonds are open, but in this case $k_N^0(\omega) = 1$ so actually this term is q.

We thus define

$$\hat{Z}_N^0(p, q) = \sum_{\omega \in \Omega_N^0} \lambda^{|C(\omega_N)|} q^{k_N^0(\omega) - 1} \tag{4.6}$$

whence

$$q \hat{Z}_N^0(p, q) = \bar{Z}_N^0(p, q) \tag{4.7}$$

in such a way that $\hat{Z}_N^0(p, q)$ can be interpreted as a partition function with term equal to 1 corresponding to the configuration in which all edges are open.

Now, by definition we can write

$$\phi_{p,q,0}^{\mathrm{f},N}(X) = \frac{1}{\hat{Z}_N^0(p, q)} \sum_{\substack{\omega \in \Omega_N^{\xi}:\ \exists g \in \mathcal{A}_N:\ E_g \in O(\omega) \\ X \subset V_g,\ V_g \cap\ \partial_v^{\rm int} \Lambda_N = \emptyset}} \lambda^{|C(\omega_N)|} q^{k_N^0(\omega) - 1}$$

We have now to write the explicit expression of $k_N^0(\omega)$. In this case we have to count the fences in the set $C(\omega_N) \cup \partial_e \Lambda_N \equiv \bar{C}(\omega_N)$, and therefore we allow fences $\bar{\gamma}$ such that $\bar{\gamma} \cap \partial_e \Lambda_N \neq \emptyset$; in the latter case the set $g \equiv \bar{\gamma} - \partial_e \Lambda_N$ will be called from now on *wall*. Observe that a wall in $\mathbb{E}_N$ is a dual animal. The number $k_N^0(\omega)$ is then simply

$$k_N^0(\omega) = n_{\bar{C}(\omega_N)}$$

Let us define for a given $S \in \mathcal{E}_N$

$$\tilde{n}_S = \begin{cases} n_S & \text{if } S \cup \partial_e \Lambda_N \notin \mathcal{E} \\ n_{S \cup \partial_e \Lambda_N} - 1 & \text{if } S \cup \partial_e \Lambda_N \in \mathcal{E} \end{cases} \tag{4.8}$$

and its activity $\rho^\xi(S)$ as follows

$$\rho^\xi(S) = \begin{cases} \lambda^{|S|} q^{n_S} & \text{if } \xi = 1 \\ \lambda^{|S|} q^{\tilde{n}_S} & \text{if } \xi = 0 \end{cases} \tag{4.9}$$

Note that

$$|\rho^\xi(S)| \le \max\{(|\lambda|\, q)^{|S|}, |\lambda|^{|S|}\} \tag{4.10}$$

The reason why we need to define for free boundary conditions the quantity $\tilde{n}_S$ is the following: for a fixed dual animal containing a wall, we can obtain a fence from the union of the wall and the (closed) boundary in two different ways, while we want to count the unit increasing of the number of connected components of the configuration. This is the reason of the -1 in the definition of $\tilde{n}_S$.

Define further the hard core pair potential between two dual animals S_i, S_j as

$$U(S_i, S_j) = \begin{cases} +\infty & \text{if } S_i \not\sim S_j \\ 0 & \text{otherwise,} \end{cases} \tag{4.11}$$

Use the shorthand notations

$$\mathbf{S}_n = (S_1, \dots, S_n)\,; \quad \rho^\xi(\mathbf{S}_n) \equiv \rho^\xi(S_1) \cdots \rho^\xi(S_n); \quad U(\mathbf{S}_n) = \sum_{1 \le i < j \le n} U(S_i, S_j)$$

Then define the ξ dependent (for $\xi = 0, 1$) polymer gas partition function as

$$\Psi_N^\xi(p, q) = 1 + \sum_{n \ge 1} \frac{1}{n!} \sum_{\mathbf{S}_n \in (\mathcal{E}_N)^n} \rho^\xi(\mathbf{S}_n) e^{-U(\mathbf{S}_n)} \tag{4.12}$$

where $(\mathcal{E}_N)^n$ is the n-times cartesian product of $\mathcal{E}_N$. Note that, by construction

$$\Psi^1_N(p,q) = \bar{Z}^1_N(p,q), \qquad \Psi^0_N(p,q) = \hat{Z}^0_N(p,q) \tag{4.13}$$

and also

$$\phi^{\mathrm{f},N}_{p,q,\xi}(X) = \frac{1}{\Psi^\xi_N(p,q)} \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{\mathbf{S}_n \in (\mathcal{E}_N)^n \\ \mathbf{S}_n \odot X}} \rho^\xi(\mathbf{S}_n) e^{-U(\mathbf{S}_n)} \tag{4.14}$$

where condition $\mathbf{S}_n \odot X$ on the sum above means that there must exist a fence $\gamma \subset \cup_{i=1}^n S_i$ such that $\gamma \odot X$ and the set $\bar{E}_\gamma \cap [\cup_{i=1}^n S_i]$ does not contains fences γ' such that $\gamma' \otimes X$ (here $\bar{E}_\gamma = \gamma \cup E_\gamma$).

We now rewrite the ratio (4.14) (between two finite sums) as a series. We follow the ideas developed in refs. 2 and 3 for $\mathbb{Z}^d$. So we will define objects more general than dual animals which will be called polymers.

Definition 4.10. *Let $X \subset \mathbb{V}$ finite, a set $P \subset \mathbb{E}$ is called X-R-connected if $P = \cup_{i=1}^k S_i$ with $k \geq 1$ and the following holds: for all $i = 1, 2, \ldots, k$ $S_i \in \mathcal{E}$; for all $i, j = 1, 2, \ldots, k$, $S_i \sim S_j$ and each S_i contains a fence γ_i such that $\gamma_i \odot Y$ for some non empty $Y \subset X$.*

We will denote by Π^X the set of all X-R-connected sets in $\mathbb{E}$ and by Π^X_N the set of all X-R-connected sets in $\mathbb{E}_N$. We will also put $\mathcal{E}^X = \mathcal{E} \cup \Pi^X$ and $\mathcal{E}^X_N = \mathcal{E}_N \cup \Pi^X_N$.

Definition 4.11. *A set $P \in \mathcal{E}^X$ will be called a X-polymer (or simply polymer when it is clear from the contest). We will say that two polymers $P_i \in \mathcal{E}^X$ and $P_j \in \mathcal{E}^X$ are compatible, and we write $P_i \approx P_j$, if $P_i \cup P_j \notin \mathcal{E}^X$; viceversa, $P_i \in \mathcal{E}^X$ and $P_j \in \mathcal{E}^X$ are incompatible, and we write $P_i \not\approx P_j$, if $P_i \cup P_j \in \mathcal{E}^X$.*

Note that if $P \in \Pi^X$ and $P' \in \Pi^X$ then necessarily $P \not\approx P'$.

If $P \in \Pi^X$ and $P = \cup_{i=1}^k S_i$ with $k \geq 2$ we define the activity of the polymer P as $\rho^\xi(P) = \prod_{i=1}^k \rho^\xi(S_i)$. Define further the hard core pair potential between two polymers P_i, P_j as

$$\tilde{U}(P_i, P_j) = \begin{cases} +\infty & \text{if } P_i \not\approx P_j \\ 0 & \text{otherwise,} \end{cases} \tag{4.15}$$

Again, we use the shorthand notations

$$\mathbf{P}_n = (P_1, \ldots, P_n)\,; \quad \rho^\xi(\mathbf{P}_n) \equiv \rho^\xi(P_1)\cdots\rho^\xi(P_n); \quad \tilde{U}(\mathbf{P}_n) = \sum_{1\leq i<j\leq n} \tilde{U}(P_i, P_j)$$

Then, the r.h.s. of (4.14) can be rewritten as

$$\phi^{\mathrm{f},N}_{p,q,\xi}(X) = \frac{1}{\Psi^{\xi}_{N}(p,q)} \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{\mathbf{P}_n \in (\mathcal{E}^X_N)^n \\ \exists! i\in \mathrm{I}_n:\ P_i \odot X}} \rho^{\xi}(\mathbf{P}_n) e^{-\tilde{U}(\mathbf{P}_n)} \tag{4.16}$$

with the hard core polymer gas partition function given by

$$\Psi^{\xi}_{N}(p,q) = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\mathbf{P}_n \in (\mathcal{E}^X_N)^n} \rho^{\xi}(\mathbf{P}_n) e^{-\tilde{U}(\mathbf{P}_n)}$$

Define now, for $\alpha \in \mathbb{R}$ and $P \in \mathcal{E}^X$

$$\rho^{\xi}_{\alpha}(P) = \begin{cases} (1+\alpha)\rho^{\xi}(P) & \text{if } P \in \Pi^X \text{ and } P \odot X \\ \\ \rho^{\xi}(P) & \text{otherwise} \end{cases}$$

then

$$\phi^{\mathrm{f}}_{p,q,\xi,N}(X) = \frac{d}{d\alpha} \ln \Psi^{\xi}_{N}(p,q,\alpha)\Big|_{\alpha=0} \tag{4.17}$$

where

$$\Psi^{\xi}_{N}(p,q,\alpha) = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\mathbf{P}_n \in (\mathcal{E}^X_N)^n} \rho^{\xi}_{\alpha}(\mathbf{P}_n) e^{-\tilde{U}(\mathbf{P}_n)} \tag{4.18}$$

where of course $\rho^{\xi}_{\alpha}(\mathbf{P}_n) = \rho^{\xi}_{\alpha}(P_1)\cdots\rho^{\xi}_{\alpha}(P_n)$.

The formal power series for $\ln \Psi^{\xi}_{N}(p,q,\alpha)$ is given by

$$\ln \Psi^{\xi}_{N}(p,q,\alpha) = \sum_{n\geq 1} \frac{1}{n!} \sum_{\mathbf{P}_n \in (\mathcal{E}^X_N)^n} \Phi^T(\mathbf{P}_n)\rho^{\xi}_{\alpha}(\mathbf{P}_n) \tag{4.19}$$

where here *the Ursell factor* $\Phi^T(\mathbf{P}_n)$ is defined as in (3.24) with $\tilde{U}$ in place of U:

$$\Phi^T(\mathbf{P}_n) = \begin{cases} \displaystyle\sum_{\substack{E\in P_2(\mathrm{I}_n) \\ (\mathrm{I}_n,E)\in\mathcal{G}_n}} \prod_{\{i,j\}\in E} (e^{-\tilde{U}(P_i,P_j)} - 1) & \text{if } n \geq 2 \\ \\ 1 & \text{if } n = 1. \end{cases} \tag{4.20}$$

where we recall that $\mathcal{G}_n$ denotes the set of all connected graphs with vertex set I_n. Thus, inserting (4.19) in (4.17) we have an explicit (formal) expansion for $\phi^{\mathrm{f},N}_{p,q,\xi}(X)$ given by

$$\phi^{\mathrm{f},N}_{p,q,\xi}(X) = \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{\mathbf{P}_n \in (\mathcal{E}^X_N)^n \\ \exists i\in \mathrm{I}_n:\ P_i \odot X}} k(\mathbf{P}_n)\Phi^T(\mathbf{P}_n)\rho^{\xi}(\mathbf{P}_n) \tag{4.21}$$

where again $k(\mathbf{P}_n) = |\{i \in \mathrm{I}_n : P_i \odot X\}|$.

We also define

$$\phi^{\mathrm{f}}_{p,q}(X) = \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{(\mathbf{P}_n \in (\mathcal{E}^X)^n \\ \exists i \in \mathrm{I}_n:\ P_i \odot X}} k(\mathbf{P}_n)\Phi^T(\mathbf{P}_n)\rho(\mathbf{P}_n) \tag{4.22}$$

where

$$\rho(\mathbf{P}_n) = \rho(P_1)\cdots\rho(P_n) \tag{4.23}$$

which, as we will see, represents an absolutely convergent expansion for small $1 - p$ for the infinite volume finite connectivity function.

4.3. Proof of Theorem 4.4

Now again by standard polymer expansion (4.21) and (4.22) are absolute convergent series provided the Kotecky and Preiss condition is satisfied. The Kotecky-Preiss condition is now

$$\sum_{n\geq n_0} \phi_n(\rho)e^{an} < a \tag{4.24}$$

where $n_0 = \min_{S\in\mathcal{E}}\{|S|\}$ and

$$\phi_n(\rho) = \sup_{e\in\mathbb{E}} \sum_{\substack{S\in\mathcal{E} \\ e\in S,\ |S|=n}} |\rho(S)| + \sum_{\substack{P\in\Pi^X \\ |P|=n}} |\rho(P)|$$

It is easy to estimate $\phi_n(\rho)$. We have

$$\sup_{e\in\mathbb{E}} \sum_{\substack{S\in\mathcal{E} \\ e\in S,\ |S|=n}} |\rho(S)| + \sum_{\substack{P\in\Pi^X \\ |P|=n}} |\rho(P)| \leq \tilde{\varepsilon}^n \left[\sup_{e\in\mathbb{E}} \sum_{\substack{S\in\mathcal{E} \\ e\in S,\ |S|=n}} 1 + \sum_{P\in\Pi^X:\,|P|=n} 1 \right]$$

where, for any λ complex and any $q > 0$

$$\tilde{\varepsilon} = \max\{|\lambda| q, |\lambda|\} \tag{4.25}$$

To get the estimates $\sum_{S\in\mathcal{E},\ e\in S,\ |S|=n} 1 \leq A_d^n$, and $\sum_{P\in\Pi^X:\,|P|=n} 1 \leq B_d^n$ we can proceed as in refs. 2 and 3. Thus, for $C_d = A_d + B_d$

$$\phi_n(\rho) \leq (C_d\tilde{\varepsilon})^n \tag{4.26}$$

Inserting (4.26) into (4.24) we have that the Kotecky-Preiss condition is satisfied if e.g. $C_d\tilde{\varepsilon} < 1/5$ Hence recalling (4.25), we obtain that the series (4.21) and (4.22) are absolutely convergent at least for all λ in the disk $|\lambda| \leq \bar{r}_q$ with

$$\bar{r}_q = \min\left\{\frac{1}{5C_d q}\ ,\ \frac{1}{5C_d}\right\} \tag{4.27}$$

Now we prove the following lemma which concludes the proof of Theorem 4.4

Lemma 4.12. *For any fixed $q > 0$ and p such that $2e\delta < 1$, and $\xi = 0, 1$*

$$\lim_{N\to\infty} \phi^{\mathrm{f},N}_{p,q,\xi}(X) = \phi^{\mathrm{f}}_{p,q}(X)$$

where $\phi^{\mathrm{f}}_{p,q}(X)$ is the function defined in (4.22).

Proof. We will consider only the case $\xi = 0$, which is the less trivial one.

$$| \phi^{\mathrm{f}}_{p,q}(X) - \phi^{\mathrm{f},N}_{p,q,\xi=0}(X) |$$

$$\leq \left| \sum_{n\geq 1} \frac{1}{n!} \left[\sum_{\substack{\mathbf{P}_n \in (\mathcal{E}^X)^n \\ \exists i \in \mathrm{I}_n:\ P_i \odot X}} k(\mathbf{P}_n)\Phi^T(\mathbf{P}_n)\rho(\mathbf{P}_n) - \sum_{\substack{\mathbf{P}_n \in (\mathcal{E}^X_N)^n \\ \exists i \in \mathrm{I}_n:\ P_i \odot X}} k(\mathbf{P}_n)\Phi^T(\mathbf{P}_n)\rho^0(\mathbf{P}_n) \right] \right|$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{\mathbf{P}_n \in (\mathcal{E}^X)^n \\ \exists i \in \mathrm{I}_n: P_i \odot X \\ \exists j \in \mathrm{I}_n:\ P_j \not\subset \mathbb{E}_N}} k(\mathbf{P}_n) \mid \Phi^T(\mathbf{P}_n) \mid\mid \rho(\mathbf{P}_n) \mid$$

$$+2 \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{\mathbf{P}_n \in (\mathcal{E}^X_N)^n \\ \exists i \in \mathrm{I}_n:\ P_i \odot X \\ \exists j \in \mathrm{I}_n:\ P_j \text{ contains a wall}}} k(\mathbf{P}_n) \mid \Phi^T(\mathbf{P}_n) \mid\mid \rho(\mathbf{P}_n) \mid$$

where we have used that $| \rho^0(\mathbf{P}_n) - \rho(\mathbf{P}_n) | \leq 2 \mid \rho(\mathbf{P}_n) \mid$, due to the bound independent on ξ (4.10).

Now, for $5C_d\tilde{\varepsilon} < 1$, the two series above are absolutely convergent. Consider the first term of the r.h.s. of this inequality. Let us split this term in two series as follows

$$\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{\mathbf{P}_n \in (\mathcal{E}^X)^n \\ \exists i \in \mathrm{I}_n:\ P_i \odot X \\ \exists j \in \mathrm{I}_n:\ P_j \not\subset \mathbb{E}_N}} k(\mathbf{P}_n) \mid \Phi^T(\mathbf{P}_n) \mid\mid \rho(\mathbf{P}_n) \mid = A_1 + A_2$$

with

$$A_1 = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{\mathbf{P}_n \in (\mathcal{E}^X)^n \\ \exists i \in \mathrm{I}_n:\ P_i \odot X \\ \exists j \in \mathrm{I}_n:\ P_j \not\subset \mathbb{E}_N,\ P_i \neq P_j}} k(\mathbf{P}_n) \mid \Phi^T(\mathbf{P}_n) \mid\mid \rho(\mathbf{P}_n) \mid$$

and

$$A_2 = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{\mathbf{P}_n \in (\mathcal{E}^X)^n \\ \exists i \in \mathrm{I}_n:\ P_i \odot X \\ \exists j \in \mathrm{I}_n:\ P_j \not\subset \mathbb{E}_N,\ P_i = P_j}} k(\mathbf{P}_n) \mid \Phi^T(\mathbf{P}_n) \mid\mid \rho(\mathbf{P}_n) \mid$$

The factor A_1 admits the bound $A_1 \leq \mathrm{Const}(C_d \tilde{\varepsilon})^{n_0}$ where the lowest order n_0 is

$$n_0 = \min_{\substack{\mathbf{P}_n \in (\mathcal{E}^X)^n,\ G(\mathbf{P}_n) \in \mathcal{G}_n \\ \exists i \in \mathrm{I}_n:\ P_i \odot X \\ \exists j \in \mathrm{I}_n:\ P_j \not\subset \mathbb{E}_N,\ P_i \neq P_j}} \left\{ \sum_{i=1}^{n} \mid P_i \mid \right\}$$

where the condition $G(\mathbf{P}_n) \in \mathcal{G}_n$ is due the presence the factor $\Phi^T(\mathbf{P}_n)$. Hence

$$n_0 \geq \min_{\substack{P_1 \odot X,\ P_2 \in \mathcal{E}^X \\ P_2 \not\subset \mathbb{E}_N, P_1 \neq P_2}} \{\mid P_1 \mid + \mid P_2 \mid + d(P_1, P_2)\}$$

where $d(P_1, P_2) = \min\{\mid x - y \mid : \{x\} \cap P_1 \neq 0,\ \{y\} \cap P_2 \neq 0\}$. Then, in the worst of hypothesis, n_0 is at least

$$n_0 \geq \min_{\substack{\gamma \in \Gamma_{\mathbb{G}},\ \gamma \odot X, \\ S \in \mathcal{E}_{\mathbb{G}},\ S \not\subset \mathbb{E}_N \\ \gamma \neq S}} \{\mid \gamma \mid + \mid S \mid + d(\gamma, S)\}$$

It is now easy to see that the r.h.s. of inequality above is a divergent quantity when $N \to \infty$.

So we have shown that $A_1 \to 0$ as $N \to \infty$.

Concerning A_2 we have similarly

$$A_2 \leq \mathrm{Const}'(C_d \tilde{\varepsilon})^{n'_0}$$

where now

$$n'_0 = \min_{\substack{\mathbf{P}_n \in (\mathcal{E}_{\mathbb{G}}^X)^n \\ \exists i \in \mathrm{I}_n:\ P_i \odot X \\ \exists j \in \mathrm{I}_n:\ P_j \not\subset \mathbb{E}_N,\ P_i = P_j}} \left\{ \sum_{i=1}^{n} \mid P_i \mid \right\} \geq \min_{\substack{P \odot X \\ P \not\subset \mathbb{E}_N}} \{\mid P \mid\}$$

this can be easily bounded from below as

$$n'_0 \geq \min_{\substack{\gamma \in \Gamma_{\mathbb{G}}:\ \gamma \odot X \\ P \not\subset \mathbb{E}_N}} \{\mid \gamma \mid\}$$

Similarly to the previous case, we have that the r.h.s. of the inequality above has to diverge when $N \to \infty$. □

To prove part a of Theorem 4.4, we have just to replicate, *mutatis mutandis*, the same reasoning developed at the end of Section 4.3. Finally, concerning the upper bound for the exponential decay of connectivities, we immediately obtain from (4.14),

$$\phi^{\mathrm{f}}_{p,q,\xi}(X) = \frac{1}{\Psi^{\xi}_{N}(p,q)} \sum_{n \geq 1} \frac{1}{n!} \sum_{\substack{\mathbf{S}_n \in (\mathcal{E}_N)^n \\ \mathbf{S}_n \odot X}} \rho^{\xi}(\mathbf{S}_n) e^{-U(\mathbf{S}_n)}$$

$$\leq \sum_{\substack{\mathbf{S} \in \mathcal{E} \\ \mathcal{E} \odot X}} \rho^{\xi}(S) \leq C'_d (B^d \lambda)^{\mathrm{fen}(X)} d$$

thus the inverse correlation length behaves as $2(d-1) \mid \ln(1-p) \mid + O(1)$ for small p.

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Note Added

Some days before the submission of the paper we were aware about the interesting result on uniqueness of the measure of the Random Cluster Model on $\mathbb{Z}^d$ in the two regions p small and $1 - p$ small for $q < 1$ contained in the new monograph by G. Grimmett to be published by Springer.[7]

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